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Deconfining phase transition in QCD_4 and QED_4 at finite temperature

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Abstract: We investigate the deconfining phase transition in QCD_4 and QED_4 at finite temperature using a perturbative deformation of topological quantum field theory (TQFT). A modified maximal abelian gauge (MAG) is utilized in the analysis. In this case, we can derive the linear potential studying the 2D theory through Parisi-Sourlas (PS) dimensional reduction. The mechanism of deconfining phase transition is proposed. It is geometrical to discuss the thermal effect on the linear potential. All we have to do is to investigate the behavior of topological objects as such instantons and vortices on a cylinder. This is the great advantage of our scenario. This mechanism is also applied in the case of QED_4. The phase structure at the high temperature of QED is investigated using the Coulomb potential on a cylinder. It coincides with the result in the lattice compact U(1) gauge theory. Also, QCD with MAG has the property called the abelian dominance, which enables us to discuss the deconfinement of QCD_4.

Keywords: Topological Field Theories, Solitons Monopoles and Instantons, Confinement, QCD.

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Contents

1. Introduction

Quark confinement is one of main problems in QCD. This phenomenon is realized at least in the low energy region (IR region) or at low temperature. However, due to asymptotic freedom the coupling constant becomes large in the IR region, where perturbation theory would not be reliable and applicable. Therefore, quark confinement should be explained from non-perturbative aspects of QCD. This phenomenon could be explained well using the dual super-conductor vacuum scenario based on monopole condensation [1]. However, this scenario is not sufficient and many ideas have been still proposed. One of the recent scenarios, which explains the quark confinement in "*continuum*" QCD, is based on using a perturbative deformation of topological quantum field theory (TQFT) [2]–[4]. This scenario describes the quark confinement very well. In particular, if we choose a modified maximal abelian gauge (MAG), which is a kind of partial gauge fixings [5], then the linear potential between a static quark and antiquark appears [2]. This means the quark confinement in the Wilson criterion [6].

The properties of QCD medium at finite temperature have been the subject of the intense study. It undergoes a drastic change as the temperature increases. It is believed that confined quarks and gluons are liberated from the certain temperature and the system is deconfined. That is, deconfining phase transition should be caused. For high temperature a characteristic energy of quarks and gluons traveling through the medium is high, and the effective coupling is small. The system presents the quark-gluon plasma (QGP). Since the effective coupling is small, we can use the perturbation theory and the perturbative calculation of its self-energy leads to the thermal mass of gluons and quarks. These mean the Debye screening effect as in the usual plasma.

In this paper, we study the deconfining phase transition of the finite temperature QCD_4 (i.e. thermal QCD) with MAG. In a previous paper [7], we have investigated the difference of the thermal phase transition mechanism between Lorentz type gauge fixing and MAG. In both cases the TQFT sector can have the phase structure. In Lorentz type gauge fixing this structure is essential to the deconfinement of full QCD_4, while in the MAG the phase structure of TQFT sector cannot have relevance in full QCD_4. Therefore, it has been unclear how we can explain the deconfinement phase transition. In this paper, we propose the deconfinement scenario in thermal QCD_4 with MAG, in which all we have to do is to investigate the behavior of topological objects such as instantons and vortices on a cylinder. For example, in an SU(2) QCD_4 we need to consider the instantons of 2D O(3) non-linear sigma model (NLSM_2 or $\mathbb{C}P^1$ model) on a cylinder [8]. However, this theory has asymptotic freedom and their instanton solution has the size parameter. Therefore the treatment seems rather difficult because the usual dilute gas sum ansatz is reliable in the very restricted region. Hence we concretely argue the deconfinement scenario in compact QED_4 in this paper. In this case, the TQFT sector becomes an O(2) NLSM_2 and only to consider their vortices on a cylinder, which have no issues as the above. We also can investigate the behavior at high temperature using the Coulomb potential on a cylinder. Moreover the result can be applied to the abelian projected effective theory, which is an abelian gauge theory with asymptotic freedom. Finally some prospects and issues in studying QCD_4 are also discussed.

Our paper is organized as follows. In section 2, we review the method of a perturbative deformation of the TQFT at zero temperature. We choose a modified MAG, which have some additional terms and the gauge fixing term has an OSp(4|2) symmetry. Quark confinement could be explained analyzing TQFT sector, in which the non-perturbative information of confinement has been encoded. This sector is equivalent to a certain 2D theory through Parisi-Sourlas dimensional reduction mechanism [9], which is caused due to an OSp(4|2) symmetry. We mainly investigate the case of $G = \mathrm{SU}(2)$, in which the TQFT sector becomes an O(3) NLSM on a plane (zero temperature). It has also instanton solutions and the linear potential between quark-antiquark pair could be induced by the instanton effect. In section 3 we consider the finite temperature case. The evaluation of the rectangular Wilson loop at zero temperature become the evaluation of two Polyakov loops' correlator. Therefore, we can investigate the phase transition from a viewpoint of 2D instantons on a cylinder. It is expected that the main role in the deconfinement phase transition is determined by the behavior of these. In section 4, we investigate the phase structure of compact QED$_4$. The method of a perturbative deformation can also be applied to QED$_4$ and it is known that zero temperature compact QED$_4$ has confining phase at large coupling [10]. In this case, the TQFT sector is an O(2) NLSM$_2$ on a cylinder. Of course, at zero temperature it is an O(2) NLSM$_2$ on 2-plane, and a confining potential is induced by the Coulomb gas of vortices. The phase transition is described by the celebrated Berezinskii-Kosterlitz-Thouless (BKT) phase transition [11]. It is natural extension to consider the Coulomb gas on a cylinder. Thus we can investigate the high temperature region of QED$_4$. We conclude in this case that thermal effect shifts the value of the coupling in which confining phase transition is caused. Moreover, we apply the result of compact QED$_4$ to the scenario in [12] by using the abelian dominance. We could explain the deconfinement which is induced by the thermal effect. Section 5 is devoted to conclusions and discussions.

2. QCD$_4$ as perturbative deformation of TQFT

2.1 Setup

In this section, we review the method of a perturbative deformation of TQFT, which is discussed in refs. [2, 3]. Firstly, we consider an SU(N) QCD at zero temperature in the 3+1 dimensional Minkowskian space-time. The modification in the case of the finite temperature system is denoted latter. We don't include matter fields, that is, consider the gluodynamics. The action for the gauge group $G = \mathrm{SU}(N)$ is

$$S = -\frac{1}{2g^2}\int d^4x \mathrm{Tr}_G F_{\mu\nu}F^{\mu\nu} \,. \tag{2.1}$$

where $F_{\mu\nu}$ is a field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

and SU(N) generators T^A, $(A = 1, \cdots, N^2-1)$, which are hermite and traceless, are normalized as

$$\mathrm{Tr}_G(T^A T^B) = \frac{1}{2}\delta^{AB} \,.$$

In order to construct the quantum field theory from the classical action (2.1), it is adequate to use the BRST quantization. Incorporating the (anti-) FP ghost field $C(\bar{C})$ and the auxiliary field B, we can construct the BRST transformation $\delta_{\rm B}$

$$\begin{aligned} \delta_{\rm B} A_\mu &= D_\mu[A]C\,, \qquad & \delta_{\rm B} C &= iC^2\,, \\ \delta_{\rm B}\bar{C} &= iB\,, \qquad & \delta_{\rm B} B &= 0\,, \end{aligned} \tag{2.2}$$

where $D_\mu[A]$ is the covariant derivative given by

$$D_\mu[A] = \partial_\mu - i[A_\mu,\ \ ]\,.$$

The gauge fixing term can be constructed from the BRST transformation $\delta_{\rm B}$ as

$$S_{\rm GF+FP} = -i\delta_{\rm B}\int d^4x G_{\rm GF+FP}[A_\mu, C, \bar{C}, B]\,, \tag{2.3}$$

where $G_{\rm GF+FP}[A_\mu, C, \bar{C}, B]$ is determined by the gauge fixing condition . If we perform the gauge fixing in the Lorentz gauge in usual manners, then we have

$$G_{\rm GF+FP} = {\rm Tr}_G\left[\bar{C}(\partial_\mu A^\mu + \frac{\alpha}{2}B)\right] = \bar{C}^A\left(\partial_\mu A^{A\mu} + \frac{\alpha}{2}B^A\right), \tag{2.4}$$

where α is a gauge parameter.

In this paper, we choose the modified MAG[1]

$$G_{\rm GF+FP}[A_\mu, C, \bar{C}, B] = \bar{\delta}_{\rm B}{\rm Tr}_{G/H}\left[A_\mu A^\mu + 2iC\bar{C}\right] = \bar{\delta}_{\rm B}\left[\frac{1}{2}A^a_\mu A^{a\mu} + iC^a\bar{C}^a\right], \tag{2.5}$$

where H is the maximal abelian subgroup, which for example is ${\rm U}(1)^{N-1}$ in $G = {\rm SU}(N)$, the subscript a denotes non-diagonal generators and $\bar{\delta}_{\rm B}$ is the anti-BRST transformation:

$$\begin{aligned} \bar{\delta}_{\rm B} A_\mu &= D_\mu[A]\bar{C}\,, \qquad \bar{\delta}_{\rm B} C = i\bar{B}\,, \\ \bar{\delta}_{\rm B}\bar{C} &= i\bar{C}^2\,, \qquad \bar{\delta}_{\rm B}\bar{B} = 0\,, \qquad B + \bar{B} = \{C, \bar{C}\}\,. \end{aligned} \tag{2.6}$$

This gauge is MAG with some additional terms. In this case, the gauge fixing term has a special symmetry, OSp(4|2) symmetry, which is very powerful to investigate the linear potential between a quark-antiquark pair.

Note that monopoles also exist in MAG. This existence is ensured by the fact that the homotopy group $\pi_2(G/H) = \pi_1(H)$ is non-trivial. These can not exist in the Lorentz type gauge.

Also, modified Lorentz gauge

$$G_{\rm GF+FP}[A_\mu, C, \bar{C}, B] = \bar{\delta}_{\rm B}{\rm Tr}_G\left[A_\mu A^\mu + 2iC\bar{C}\right] = \bar{\delta}_{\rm B}\left[\frac{1}{2}A^A_\mu A^{A\mu} + iC^A\bar{C}^A\right],$$

is discussed in ref. [3]. The comparison between MAG type and Lorentz type in $G = {\rm SU}(2)$ is shown in figure 1.

[1] This choice is rather specific, and more general gauge fixing given by

$$G_{\rm GF+FP} = {\rm Tr}_{G/H}\left[A_\mu A^\mu - i\alpha C\bar{C}\right]$$

is also possible. In this case, the interesting phenomenon, which is discussed in ref. [13], is caused.

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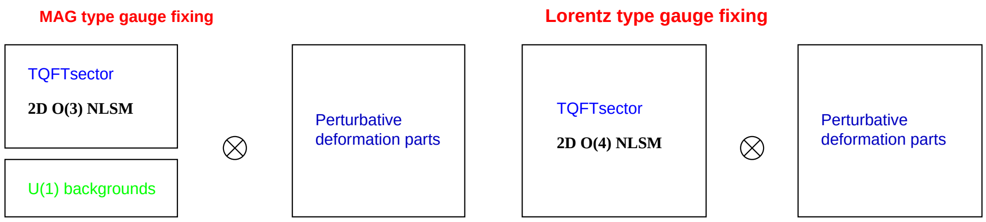

Figure 1: The comparison between Lorentz type gauge and MAG type gauge in $G =$ SU(2). The phase structure of the TQFT sector is preserved in the case of the Feynman type gauge fixing. In MAG, however, it is screened by the perturbative fluctuation and its information is encoded in the U(1) backgrounds.

2.2 Decomposition into topological trivial and non-trivial sectors

In this section, we decompose the action of QCD into a topological trivial sector and a non-trivial sector which is described by a topological quantum field theory (TQFT), called a TQFT sector. This method was proposed in ref. [3] in the context of Lorentz type gauge and extended to the MAG in ref. [2]. Thus, we may recapitulate QCD as a perturbative deformation from a TQFT sector. In this method, the information on the non-perturbative phenomena of QCD, in particular quark confinement and phase structure is encoded in the TQFT sector.

We begin with the following decomposition

$$A_\mu = U V_\mu U^\dagger + iU\partial_\mu U^\dagger \,, \tag{2.7}$$

where we define Ω_μ as

$$\Omega_\mu \equiv iU\partial_\mu U^\dagger \,.$$

We assume that A_μ is given by a finite rotation of V_μ. Here Ω_μ is composed of compact degrees of freedom U alone, but $UV_\mu U^\dagger$ is not compact. In below, we assume that the non-compact gauge field variable V_μ does not have topologically non-trivial configuration and all topologically non-trivial configurations come from the compact gauge group variable U alone. As suggested in ref. [16], it is expected that the perturbation-theoretical study has nothing to do with confinement and the real deep reason for the confinement is encoded into the topological structure of the gauge group.

Secondly, we introduce the FP determinant $\Delta_{\rm FP}[A]$ defined by

$$\Delta_{\rm FP}[A]^{-1} \equiv \int [dU] \prod_{x,A} \delta\left(\partial^\mu A_\mu^{U^{-1}}(x)\right) , \tag{2.8}$$

where $[dU]$ is the gauge invariant Haar measure and so this determinant is invariant under the gauge transformation,

$$\Delta_{\rm FP}[A] = \Delta_{\rm FP}[A^{U^{-1}}] \,.$$

Then we rewrite the unit as follows,

$$\begin{aligned}1 &= \Delta_{\rm FP}[A] \int [dU] \prod_{x,A} \delta\left(\partial^\mu A_\mu^{U^{-1}}\right) \\ &= \Delta_{\rm FP}[A^{U^{-1}}] \int [dU] \prod_{x,A} \delta\left(\partial^\mu A_\mu^{U^{-1}}\right) \\ &= \Delta_{\rm FP}[V] \int [dU] \prod_{x,A} \delta\left(\partial^\mu V_\mu^A\right) \\ &= \int [dU][d\gamma][d\bar\gamma][d\beta] \exp\left[i \int d^4x 2\mathrm{Tr}_G (\beta \partial^\mu V_\mu + i\bar\gamma \partial^\mu D_\mu[V]\gamma)\right] . \end{aligned} \tag{2.9}$$

Here, we define the new BRST transformation $\tilde\delta_{\rm B}$ as

$$\begin{aligned} \tilde\delta_{\rm B} V_\mu &= D_\mu[V]\gamma\,, \qquad & \tilde\delta_{\rm B}\gamma &= i\gamma^2\,, \\ \tilde\delta_{\rm B}\bar\gamma &= i\beta\,, & \tilde\delta_{\rm B}\beta &= 0\,. \end{aligned} \tag{2.10}$$

By the use of this $\tilde\delta_{\rm B}$, eq. (2.9) can be rewritten as

$$1 = \int [dU][d\gamma][d\bar\gamma][d\beta] \exp\left[i \int d^4x \left(-i\tilde\delta_{\rm B} \tilde G_{\rm GF+FP}[V_\mu, \gamma, \bar\gamma, \beta]\right)\right], \tag{2.11}$$

where $\tilde G_{\rm GF+FP}[V_\mu, \gamma, \bar\gamma, \beta]$ is written as

$$\tilde G_{\rm GF+FP}[V_\mu, \gamma, \bar\gamma, \beta] \equiv 2\mathrm{Tr}_G(\bar\gamma \partial^\mu V_\mu)\,. \tag{2.12}$$

When we insert eq. (2.11) into the partition function, it is rewritten as

$$\begin{aligned} Z[J] = \int [dU][dC][d\bar C][dB] \int [dV][d\gamma][d\gamma] \exp\Big[i \int d^4x \Big(&- \frac{1}{2g^2}\mathrm{Tr}_G\left(F_{\mu\nu}[V]F^{\mu\nu}[V]\right) - \\ - i\tilde\delta_{\rm B}\tilde G_{\rm GF+FP}[V_\mu, \gamma, \bar\gamma, \beta] - i\delta_{\rm B} G_{\rm GF+FP}[\Omega_\mu &+ UV_\mu U^\dagger, C, \bar C, B]\Big) + iS_{\rm J}\Big], \end{aligned} \tag{2.13}$$

where $S_{\rm J}$ is a source term given by

$$S_{\rm J} = \int d^4x \mathrm{Tr}_G \left[J^\mu \left(\Omega_\mu + UV_\mu U^\dagger\right) + J_{\rm c} C + J_{\bar C}\bar C + J_{\rm B} B\right] . \tag{2.14}$$

Note that the transformation law of U and V_μ under $\delta_{\rm B}$ and $\bar\delta_{\rm B}$ is expressed as

$$\begin{aligned} \delta_{\rm B} U &= iCU\,, \qquad & \bar\delta_{\rm B} U &= i\bar C U\,, \\ \delta_{\rm B} V_\mu &= 0\,, & \bar\delta_{\rm B} V_\mu &= 0\,. \end{aligned} \tag{2.15}$$

By the use of these transformation laws, we can rewrite $\delta_{\rm B} G_{\rm GF+FP}[\Omega_\mu + UV_\mu U^\dagger]$ as

$$\begin{aligned} \int d^4x \delta_{\rm B} G_{\rm GF+FP}[\Omega_\mu + UV_\mu U^\dagger, C, \bar C, B] &= \\ = \int d^4x \delta_{\rm B} G_{\rm GF+FP}[\Omega_\mu, C, \bar C, B] &+ \int d^4x \left(V_\mu^A \mathcal{M}^{A\mu}[U] + \frac{1}{2} V_\mu^A V^{B\mu} \mathcal{K}^{AB}[U]\right), \end{aligned} \tag{2.16}$$

where $\mathcal{M}_\mu^A[U]$, and $\mathcal{K}^{AB}[U]$ are defined as

$$\mathcal{M}_\mu^A[U] \equiv \delta_{\rm B}\bar{\delta}_{\rm B}\left((UT^AU^\dagger)^a\Omega_\mu^a\right), \qquad \mathcal{K}^{AB}[U] \equiv \delta_{\rm B}\bar{\delta}_{\rm B}\left((UT^AU^\dagger)^a(UT^BU^\dagger)^a\right).$$

Finally, we obtain the following expression of the partition function

$$\begin{aligned} Z[J] = &\int [dU][dC][d\bar{C}][dB]\exp\left(iS_{\rm TQFT} + iW[U;J^\mu]+\right. \\ &+i\int d^4x(J^\mu\Omega_\mu + J_{\rm C}C + J_{\bar{\rm C}}\bar{C} + J_{\rm B}B)\,, \end{aligned} \tag{2.17}$$

where $S_{\rm TQFT}$ is defined by

$$\begin{aligned} S_{\rm TQFT} &\equiv -i\delta_{\rm B}G_{\rm GF+FP}[\Omega_\mu, C, \bar{C}, B] \\ &= -i\int d^4x\delta_{\rm B}\bar{\delta}_{\rm B}{\rm Tr}_{G/H}\left[\Omega_\mu\Omega^\mu + 2iC\bar{C}\right]. \end{aligned} \tag{2.18}$$

Here, $W[U;J^\mu]$ in eq. (2.17) denotes a perturbative deformation and is defined as

$$\begin{aligned} \exp\left(iW[U;J^\mu]\right) \equiv \int [dV][d\gamma][d\bar{\gamma}][d\beta]\exp\Big(&iS_{\rm pQCD}[V_\mu,\gamma,\bar{\gamma},\beta] + \\ &+i\int d^4x\Big(V_\mu\mathcal{J}^\mu - \frac{1}{2}V_\mu^AV^{B\mu}\mathcal{K}^{AB}[U]\Big)\Big)\,, \end{aligned} \tag{2.19}$$

where $\mathcal{J}_\mu$ is the new source term redefined as

$$\mathcal{J}_\mu^A \equiv U^\dagger J_\mu^A U - \mathcal{M}_\mu^A[U]\,,$$

and $\mathcal{K}$ describes the interaction between pQCD and TQFT sectors. pQCD denotes the perturbative QCD (topological trivial sector). Here, the action $S_{\rm pQCD}$ is defined by

$$S_{\rm pQCD}[V_\mu,\gamma,\bar{\gamma},\beta] \equiv \int d^4x\left(-\frac{1}{2g^2}F_{\mu\nu}[V]F^{\mu\nu}[V] - i\tilde{\delta}_{\rm B}\tilde{G}_{\rm GF+FP}[V_\mu,\gamma,\bar{\gamma},\beta]\right). \tag{2.20}$$

Note that $\mathcal{M}_\mu^A[U]$ and $\mathcal{K}^{AB}[U]$ are interactions between pQCD and TQFT sector.

The perturbative deformation $W[U;J^\mu]$ is the generating functional of the connected Green function of V_μ, which describes the perturbative deformation sector. This should be calculated by the use of the ordinary perturbation theory with the expansion of the coupling constant g

$$\begin{aligned} iW[U;J^\mu] &\equiv \ln\left\langle\exp\left(i\int d^4x[V_\mu^A\mathcal{J}^{A\mu} - V_\mu^AV^{B\mu}\mathcal{K}^{AB}]\right)\right\rangle_{\rm pQCD} \\ &= \frac{g^2}{2}\int d^4x\int d^4y\,\langle V_\mu(x)V_\nu(y)\rangle_{\rm pQCD}^c\left(\mathcal{J}^\mu(x)\mathcal{J}^\nu(y) - \delta^4(x-y)\eta^{\mu\nu}\mathcal{K}^{AB}[U]\right) + \\ &\quad + \text{higher order of } g\,. \end{aligned} \tag{2.21}$$

Therefore, $W[U;J^\mu]$ is expressed as a power series in the coupling constant g and goes to zero as $g \to 0$. It turns out that the full QCD is reduced to the TQFT sector in the vanishing limit of coupling constant. Thus we can interpret the term $W[U;J^\mu]$ as the deformation from the TQFT sector.

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2.3 Relation between expectation values

Let us discuss below in the euclidean metric. We can define the expectation value in each sector as

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_m \rangle_{\rm TQFT} \equiv \int [dU][dC][d\bar{C}]\mathcal{O}_1 \ldots \mathcal{O}_m \exp\left(-S_{\rm TQFT}[U,C,\bar{C},B]\right), \quad (2.22)$$

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\rm pQCD} \equiv \int [dV][d\gamma][d\bar{\gamma}][d\beta]\mathcal{O}_1 \ldots \mathcal{O}_n \exp\left(-S_{\rm pQCD}[V_\mu,\gamma,\bar{\gamma},\beta]\right), \quad (2.23)$$

and reconstruct the expectation value of the full QCD_4. If the inserted operator is decomposed as $f(A) = g(V,U)h(U)$, then the full expectation value of this is expressed as

$$\langle f(A)\rangle_{\rm QCD} = \Big\langle \langle g(V,U)\rangle_{\rm pQCD}\, h(U)\Big\rangle_{\rm TQFT} = \Big\langle \langle g(V,U)h(U)\rangle_{\rm TQFT}\Big\rangle_{\rm pQCD}. \quad (2.24)$$

Thus we can obtain the full expectation value through the above expectation values in each sector. Which of expression eq. (2.24) should be utilized in calculating the expectation values depends on the case. In fact, the decomposition of the inserted operator is rather difficult in non-abelian gauge group. Our purpose is to investigate the quark confinement and so would like to evaluate a non-abelian Wilson loop

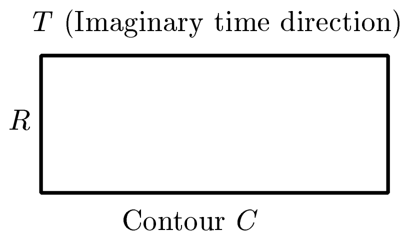

Figure 2: The rectangular Wilson loop.

$$W_C = \mathrm{Tr} P \exp\left(ie \oint_C dx^\mu A_\mu\right),$$

where the contour C is the rectangular loop as shown in figure 2.

In this case, the expectation value of non-abelian Wilson loop is mathematically decomposed as the above by the use of non-abelian Stokes theorem [14, 15]. When we consider the case of $G = \mathrm{SU}(2)$ for simplicity,[2] it could be expressed as

$$\begin{aligned}&\langle W_C[A]\rangle_{\rm QCD} = \\ &= \int d\mu(U)\left\langle \left\langle \exp\left[ie\oint_C dx^\mu \mathrm{Tr}\left(\sigma_3 U V_\mu U^\dagger\right)\right]\right\rangle_{\rm pQCD} \exp\left[ie\oint_C dx^\mu \Omega^3_\mu\right]\right\rangle_{\rm TQFT}, \end{aligned} \quad (2.25)$$

where $\Omega^3_\mu \equiv 2\mathrm{Tr}(T^3\Omega_\mu)$ and $d\mu$ is the invariant Haar measure of the coset space $\mathrm{SU}(2)/\,\mathrm{U}(1)$. When we expand eq. (2.25) perturbatively, it becomes

$$\begin{aligned}\langle W_C[A]\rangle_{\rm QCD} = \int d\mu(U)\Big\langle \exp\left[ie\oint_C dx^\mu\Omega^3_\mu\right]\Big\rangle_{\rm TQFT} - \frac{e^2}{2}\oint dx^\mu \oint dy^\nu D_{\mu\nu}(x-y)\,\times \\ \times \int d\mu(U)\Big\langle \exp\left[ie\oint_C dx^\mu\Omega^3_\mu\right] 2\mathrm{Tr}\{T^3 U(x) T^A U^\dagger(x)\}\,\times \\ \times\, 2\mathrm{Tr}\{T^3 U(y) T^A U^\dagger(y)\}\Big\rangle_{\rm TQFT} + \\ +\ \text{higher order of } e\,, \end{aligned} \quad (2.26)$$

[2] For $G = S\,\mathrm{U}(N)$, the expression is rather complicated.

where we used the following relations

$$\left\langle V_\mu^A(x)\right\rangle_{\mathrm{pQCD}} = 0\,, \quad \left\langle V_\mu^A(x)V_\nu^B(y)\right\rangle_{\mathrm{pQCD}} = \frac{\delta^{AB}\delta_{\mu\nu}}{4\pi^2|x-y|^2} \equiv \delta^{AB}D_{\mu\nu}(x-y)\,.$$

Here e is a charge of the external source and proportional to g, for example $e = g/2$ for the fundamental representation. Hence, the above expansion is about the power of g. As we will see later, the evaluation of the "*abelian*" Wilson loop

$$\exp\left[ie\oint_C dx^\mu \Omega_\mu^3\right] \tag{2.27}$$

leads to the linear potential. Note that eq. (2.26) does not imply the abelian dominance, which is well known feature in the MAG, but tell us what we should evaluate.

However, in the case of QED_4 the decomposition as the above is completely done as follows [10],

$$\langle W_C[A]\rangle_{\mathrm{QED}} = \langle W_C[\Omega]\rangle_{\mathrm{TQFT}}\,\langle W_C[V]\rangle_{\mathrm{p\,U(1)}}\;, \tag{2.28}$$

where

$$A_\mu = V_\mu + \frac{i}{g}U\partial_\mu U^\dagger\,, \qquad \Omega_\mu \equiv \frac{i}{g}U\partial_\mu U^\dagger\,.$$

This is similar with the decomposition of the partition function

$$Z = Z_{\mathrm{inst}}\cdot Z_{\mathrm{p\,U(1)}}$$

in the result of Polyakov's work [16], in which the linear potential is derived from Z_{inst} though the theory is on 3D euclidean.

2.4 PS Dimensional Reduction to 2D Theory

Parisi-Sourlas mechanism can dimensionally reduce 4D TQFT sector to 2D theory [9]. This is because we have chosen the special gauge fixing that has an OSp(4|2) symmetry. Though the 2D space can be taken arbitrarily, we should take a 2-plane at zero temperature and a cylinder at finite temperature as 2D space respectively in order to evaluate Wilson loop to derive the linear potential and investigate quark confinement . As a result, the action of TQFT sector becomes as follows,

$$S_{\mathrm{TQFT}} = \frac{2\pi}{g^2}\int d^2x \mathrm{Tr}_{G/H}\left[\Omega_\mu\Omega^\mu + 2iC\bar{C}\right]. \tag{2.29}$$

It describes a coset G/H chiral model on 2D space.

In the case of SU(2), this action can be rewritten as the action of O(3) NLSM,

$$S_{\mathrm{TQFT}} = \frac{\pi}{g^2}\int d^2x\partial_\mu\mathbf{n}\cdot\partial^\mu\mathbf{n}\,, \qquad \mathbf{n}\cdot\mathbf{n} = 1\,. \tag{2.30}$$

Here we used the Euler angle representation of an SU(2) matrix

$$
\begin{aligned}
\mathrm{U}(x) &= \exp\left(i\chi(x)\frac{\sigma_3}{2}\right)\exp\left(i\theta(x)\frac{\sigma_2}{2}\right)\exp\left(i\varphi(x)\frac{\sigma_3}{2}\right) \\
&= \begin{pmatrix} \exp\left(\frac{i}{2}(\varphi+\chi)\right)\cos\frac{\theta}{2} & \exp\left(-\frac{i}{2}(\varphi-\chi)\right)\sin\frac{\theta}{2} \\ -\exp\left(\frac{i}{2}(\varphi-\chi)\right)\sin\frac{\theta}{2} & \exp\left(-\frac{i}{2}(\varphi+\chi)\right)\cos\frac{\theta}{2} \end{pmatrix}, \qquad (2.31)
\end{aligned}
$$

and parameterized an unit length vector field as

$$
\mathbf{n}(x) \equiv \begin{pmatrix} n^1(x) \\ n^2(x) \\ n^3(x) \end{pmatrix} \equiv \begin{pmatrix} \sin\theta(x)\cos\varphi(x) \\ \sin\theta(x)\sin\varphi(x) \\ \cos\theta(x) \end{pmatrix}. \qquad (2.32)
$$

We can investigate a TQFT sector through an NLSM_2. In particular, the expectation values of both theories are given as follows,

$$
\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathrm{TQFT}_4} = \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathrm{NLSM}_2}\,. \qquad (2.33)
$$

2.5 Confinement and static potential

Here, we can evaluate the abelian Wilson loop through O(3) NLSM instantons as follows,

$$
\begin{aligned}
\langle W_C[\Omega]\rangle_{\mathrm{TQFT}_4} &\equiv \left\langle \exp\left[ie\oint_C dx^\mu \Omega_\mu^3(x)\right]\right\rangle_{\mathrm{TQFT}_4} \\
&= \left\langle \exp\left[2\pi i\left(\frac{e}{g}\right)Q_{\mathrm{NLSM}_2}\right]\right\rangle_{\mathrm{TQFT}_4} \\
&= \left\langle \exp\left[2\pi i\left(\frac{e}{g}\right)Q_{\mathrm{NLSM}_2}\right]\right\rangle_{\mathrm{NLSM}_2}, \qquad (2.34)
\end{aligned}
$$

where e is a charge of an external source, and

$$
Q_{\mathrm{NLSM}_2} = \frac{1}{8\pi}\int_S d^2x\epsilon_{\mu\nu}\mathbf{n}(x)\cdot(\partial_\mu\mathbf{n}(x)\times\partial_\nu\mathbf{n}(x))\,, \qquad (2.35)
$$

is an instanton density of an NLSM_2. Thus, we can calculate this abelian Wilson loop by the use of dilute gas approximation,[3] and the result is given as

$$
\langle W_C[\Omega]\rangle_{\mathrm{TQFT}_4} = \exp\left(-\sigma A\right). \qquad (2.36)
$$

Here $A = RT$ is the area spanned by the contour C, and σ, which is a string tension of confining string, is given by

$$
\sigma = 2B\mathrm{e}^{-S_1}\,, \qquad S_1 = \frac{8\pi}{g^2}\,, \qquad (2.37)
$$

[3] We could improve the instanton calculation including the interaction between (anti-)instantons [17].

for the fundamental representation. In eq. (2.37) S_1 is the 1-instanton action and B is the constant derived from the integration of the instanton moduli. Because the instanton size ρ is proportional to the inverse mass of off-diagonal gluons m_{A}^{-1}, B is a constant of order m_{A}^2. In order to know the precise form of B, we must determine the measure for the collective coordinate ρ. When the contribution of the perturbative deformation part is included, the full Wilson loop expectation value is written as

$$\langle W_C[A]\rangle_{\mathrm{QCD}} = \mathrm{e}^{-\sigma RT}\left[1+\left(\frac{3}{4}\right)\frac{e^2}{4\pi R}Tf(R)+\cdots\right], \tag{2.38}$$

and the full static potential between a pair of quark and antiquark is expressed by

$$V(R) = \sigma R - \left(\frac{3}{4}\right)\frac{e^2}{4\pi R}f(R)+\cdots, \tag{2.39}$$

where $f(R)$ is a certain function that behaves as $f(R)\to 1, (R\to 0)$.

Thus the linear potential is induced by the instanton effect in an O(3) NLSM_2 in the leading. This means quark confinement in the Wilson criterion. Hence we must consider the behavior of the instantons in order to study the confinement and deconfinement essentially. Note that the NLSM_2 instantons should be interpreted as the points that monopoles' current lines pierce the 2D space which has been chosen in dimensionally reducing TQFT_4 sector (for details, see ref. [2]), and so we can say that monopoles are essentially relevant to the confinement.

3. Confining phase and deconfining phase transition

3.1 Finite temperature

Now we would like to consider the finite temperature system coupled to the thermal bath. The imaginary time formalism and real time formalism [18]–[20] are well known procedure to investigate a thermal field dynamics (TFD). In both cases, gauge fields obey the boundary conditions

$$A_\mu(-i\beta,\mathbf{x}) = A_\mu(0,\mathbf{x}) \tag{3.1}$$

for an imaginary time direction.

The twisted boundary conditions for the gauge group element $\mathrm{U}(x)$

$$B_l : U(-i\beta,\mathbf{x}) = U(0,\mathbf{x})\mathrm{e}^{2\pi l i/N}, \qquad (l=0,\ldots,N-1) \tag{3.2}$$

can be imposed using the element of the center. For $l=0$ it is a periodic one.

The FP determinant at finite temperature is modified as follows,

$$1 \equiv \Delta[A]\frac{1}{N}\sum_{l=0}^{N-1}\int_{B_l}[dU]\prod_{x,A}\delta\left(\partial^\mu A_\mu^{U^{-1}}(x)\right). \tag{3.3}$$

Due to the thermal effect it is decomposed into N independent sectors. In each sector the gauge transformation obeys each boundary condition given by eq. (3.2). It is likely to consider that such decomposition should have something to do with the domain wall, which is related to the spontaneous discrete symmetry break down, such as the center symmetry $\mathbb{Z}_N$ [21]. This point remains to be unclear.

At finite temperature, the expectation value is decomposed to the sum as

$$\begin{aligned} \langle f(A)\rangle_{\mathrm{QCD}} &\equiv \frac{1}{N}\sum_{i=0}^{N-1}\Big\langle \langle g(V,U)\rangle_{\mathrm{pQCD}}\, h(U)\Big\rangle_{\mathrm{TQFT}}^{(i)} \\ &= \frac{1}{N}\sum_{i=0}^{N-1}\Big\langle \langle g(V,U)h(U)\rangle_{\mathrm{TQFT}}^{(i)}\Big\rangle_{\mathrm{pQCD}}\,. \end{aligned} \tag{3.4}$$

3.2 Boundary conditions in reduced 2D theory

Let us consider the case of $N = 2$ concretely. In $N = 2$ using PS dimensional reduction both $\mathrm{TQFT}_4^{(i)}$ $(i = 0, 1)$ sectors are described by the field $\mathbf{n}(x)$ obeying the periodic condition for imaginary time direction [7].

Boundary conditions of $\mathbf{n}(x)$ are derived from the following useful relations

$$n^A(x)T^A = U^\dagger(x)T^3U(x)\,, \quad n^A(x) = 2\mathrm{Tr}_G[U(x)T^AU^\dagger(x)T^3]\,, \quad (A = 1, 2, 3)\,. \tag{3.5}$$

We find that $\mathbf{n}(x)$ is invariant under U(1) transformation generated by T^3 and it can be rotated by generators associated with the coset SU(2)/ U(1). Also, eq. (2.29) has the following global $\mathrm{SU(2)_L}$ symmetry[4]

$$U \longrightarrow Uh\,, \qquad \forall\ h \in \mathrm{SU(2)_L}\,. \tag{3.6}$$

Then $\mathbf{n}(x)$ transforms as

$$n^AT^A \longrightarrow n^A(h^\dagger T^Ah) \equiv n'^AT^A \tag{3.7}$$

and we easily see the action of h on n^A

$$n^A \longrightarrow n'^A = \sum_{B=1}^{3} \mathrm{ad}(h^\dagger)^A{}_B n^B\,. \tag{3.8}$$

This means that $\mathbf{n}(x)$ should be transformed under SO(3) rotation but it is invariant by an action of the center $\mathbb{Z}_2$ of SU(2). By the use of eq. (3.8), boundary conditions for U(x) can be translated into that on field $\mathbf{n}(x)$

$$\begin{aligned} n^A(-i\beta,\sigma) &= \sum_{B=1}^{3}\mathrm{ad}(g^\dagger)^A{}_B n^B(0,\sigma)\,, \qquad g \in \mathbb{Z}_2 \\ &= n^A(0,\sigma)\,, \end{aligned} \tag{3.9}$$

where σ is a spatial coordinate of 2D space. Thus, $\mathbf{n}(x)$ obeys a periodic condition.

[4]In the Lorentz type gauge fixing, TQFT sector becomes 2D O(4) $\mathrm{NLSM_2}$, which has a global chiral symmetry $\mathrm{SU(2)_L \otimes SU(2)_R}$. While in MAG $\mathrm{SU(2)_L}$ symmetry only exists. Also, this system does not have instanton solutions.

In conclusion, boundary conditions of $\mathrm{U}(x)$ is irrelevant in MAG from the viewpoint of reduced 2D theory and its contribution to the full expectation value is same in each sector, i.e.

$$\langle f(A)\rangle_{\mathrm{QCD}} = \frac{1}{2}\sum_{i=0}^{1}\left\langle \langle g(V,U)h(U)\rangle^{(i)}_{\mathrm{TQFT}}\right\rangle_{\mathrm{pQCD}} \longrightarrow \left\langle \langle g(V,U)h(U)\rangle_{\mathrm{TQFT}}\right\rangle_{\mathrm{pQCD}} . \tag{3.10}$$

This point is different from the case of the Lorentz type where the field variables on a reduced 2D theory obey twisted boundary conditions in each TQFT sector.

In the case of $N \geq 3$, there are some possibilities to take the coset space, so it is nontrivial whether similar result is concluded. At least if we take the flag space $F_N \cong \mathrm{SU}(N)/\mathrm{U}(1)^{N-1}$ (maximal abelian) or the complex projective space $\mathbb{C}P^{N-1} \cong \mathrm{SU}(N)/(\mathrm{SU}(N-1)\times \mathrm{U}(1))$ as the coset, then the similar result seems to be followed [15]. Due to the U(1) factor in the coset the structure such as eq. (3.10) would be followed.

Comment on phase structure of TQFT sector. If we consider only TQFT sector without perturbative deformation part (or consider pure gauge configuration only), the twist factor becomes arbitrary element of SU(2) instead of the center, and more general boundary conditions of $\mathrm{U}(x)$ are allowed as follows,

$$B_g : \quad U(-i\beta, \mathbf{x}) = U(0,\mathbf{x})g\,, \qquad (\forall\ g \in G)\,. \tag{3.11}$$

Also, $\mathbf{n}(x)$ can obey twisted boundary conditions. In this case, TQFT sector can have the phase structure though spontaneously symmetry breaking (SSB), though it is forbidden in 2D theory by the novel Coleman-Mermin-Wagner theorem [22]. This is shown in ref. [3] by calculating the effective potential.

3.3 Polyakov loop and confinement at finite temperature

Here, using the imaginary time formalism we consider an $\mathrm{SU}(N)$ gauge theory at finite temperature, in which the order parameters for deconfining phase transition are the expectation values of the Wilson lines wrapping k times around the compact time dimensions $\tau = -ix^0$,

$$P_k(\vec{x}) = \mathrm{Tr}P\exp\left[i\int_0^{k\beta} d\tau A_\tau(\tau,\vec{x})\right], \tag{3.12}$$

where P denotes the path-ordering and $P_k(\vec{x}),\ (k\in\mathbb{Z})$ are also called Polyakov loops. The theory has a $\mathbb{Z}_N$ symmetry, which allows us to impose the twisted boundary conditions

$$U(\beta,\vec{x}) = \mathrm{e}^{2\pi li/N}U(0,\vec{x}) \qquad (l=0,\ldots,N-1) \tag{3.13}$$

for gauge group elements. Under this gauge transformation the Polyakov loop transforms as

$$P_k(\vec{x}) \longrightarrow \mathrm{e}^{2\pi kli/N}P_k(\vec{x})\,. \tag{3.14}$$

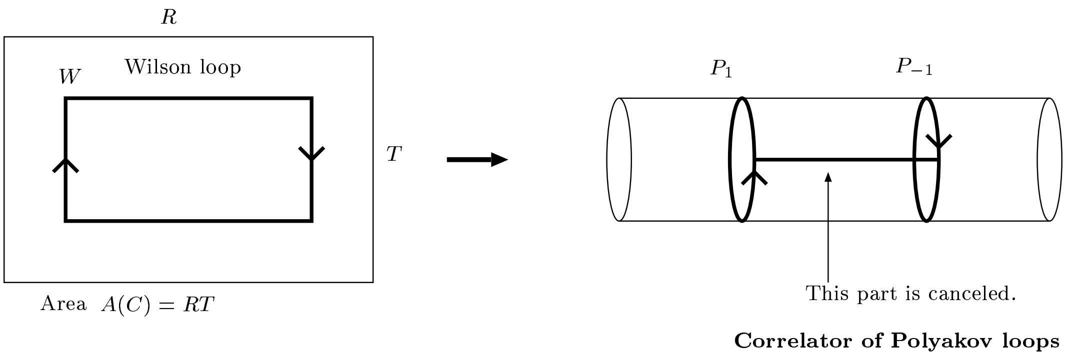

Figure 3: The correlator of Polyakov loops. The expectation value of a Wilson loop W is equivalent to a correlator of the Polyakov loops P_1 and P_{-1}.

Therefore $\langle P_k(\vec{x})\rangle = 0$ means that $\mathbb{Z}_N$ symmetry is unbroken. A Polyakov loop corresponds to an external quark source (in the fundamental representation of $\mathrm{SU}(N)$). The free energy of the system (heat bath) is increased by adding such a source. If we write this additional free energy as F_q, the expectation value of a Polyakov loop can be expressed as

$$\langle P_k(\vec{x})\rangle \sim \mathrm{e}^{-\beta F_q}\,. \tag{3.15}$$

From eq. (3.15), $\langle P_k(\vec{x})\rangle = 0$ implies that the free energy cost is infinite and an isolated quark can not exist in the theory. While if $\langle P_k(\vec{x})\rangle \neq 0$, it is finite and an isolated quark has finite energy, i.e. the theory no longer confines. In summary,

$$\begin{aligned} \langle P_k(\vec{x})\rangle = 0 \Longrightarrow \qquad & \text{confining phase and } \mathbb{Z}_N \text{ symmetry is unbroken}\,, \\ \langle P_k(\vec{x})\rangle \neq 0 \Longrightarrow \qquad & \text{deconfining phase and } \mathbb{Z}_N \text{ symmetry is broken}\,. \end{aligned}$$

Thus $\mathbb{Z}_N$ symmetry has been considered to be related to deconfinement transition. Also, $\mathbb{Z}_N$ symmetry is important to decide the order of the phase transition.

In general, QCD_4 is believed to be confined at low temperature. Therefore, the correlator of two Polyakov loops is expected to behave as

$$\begin{gathered} \langle P_k(\vec{R})P_{-k}(0)\rangle \sim \mathrm{e}^{-\beta F_{q\bar{q}}} \longrightarrow 0\,, \qquad (R = |\vec{R}| \longrightarrow \infty)\,, \\ F_{q\bar{q}} = \sigma R\,, \qquad (\sigma:\ \text{string tension}) \end{gathered} \tag{3.16}$$

at low temperature. That is, it should show the exponentially decay law. Also, eq. (3.16) implies $|\langle P\rangle|^2 = 0$, and so confinement.

Therefore we would like to evaluate the correlator of the Polyakov loops in order to study the confinement. Note that the rectangular Wilson loop at zero temperature becomes the Polyakov loops' correlator at finite temperature as shown in figure 3.

Thus,

$$\langle P_1(\vec{R})P_{-1}(0)\rangle = \langle W_C[\Omega]\rangle \tag{3.17}$$

is followed. Therefore, in the same way as the case of the zero temperature, we can derive the linear potential and obtain the following result

$$\langle P_1(\vec{R})P_{-1}(0)\rangle \sim \mathrm{e}^{-\beta\sigma R}\,. \tag{3.18}$$

Note that string tension σ depends on the temperature T as

$$\sigma = 2B(T)\mathrm{e}^{-S_1}\,, \qquad B(T) \sim T^2\,, \qquad S_1 = \frac{8\pi}{g^2(T)}\,, \tag{3.19}$$

because the instanton moduli integral is restricted by the radius of the cylinder[5] and the effective coupling depends on the temperature. Therefore the string tension depends on the temperature continuously, and decrease as the temperature increases.

Also, a perturbative deformation part would be modified by the thermal effect. In particular, we can derive the Yukawa type potential, which means the Debye screening effects as in the usual plasma instead of the Coulomb potential at the sufficiently high temperature region. In this time, we use the standard calculation in TFD and use the hard thermal loop (HTL) approximation [23]. Therefore, calculating the contribution of the gluon self-energy to its propagator at 1-loop level with HTL approximation, we obtains the

$$\langle V_\mu^a(x)V_\nu^b(y)\rangle_{\mathrm{pQCD}} = \frac{\delta_{\mu\nu}\delta^{ab}}{4\pi|x-y|}\mathrm{e}^{-m_D|x-y|}\,, \qquad m_D^2 = \frac{1}{6}g^2T^2C_A\,, \tag{3.20}$$

where m_D is the thermal gluon mass and $C_A = N$ is the group constant.

3.4 Deconfinement phase transition

The result eq. (3.18) is favorable at least in low temperature region but at high temperature region the system should be deconfined. Therefore it is expected that the string tension should vanish above a certain temperature, i.e. deconfining phase transition should be caused by the thermal effect.[6] This problem has been remarked in our previous paper [7]. Let us recall that the linear potential is induced by topological objects in our scenario. In order for the linear potential to vanish, the effect of topological objects has to be ignored. For example, a pair of topological objects needs to form the bound state and behaves as "neutral molecules", or topological objects decay. In such case, the linear potential would vanish since it is induced by non-zero topological charge in the rectangular Wilson loop. From the viewpoint of reduced 2D theory, the thermal effect is realized through the radius of the cylinder. While at low temperature the cylinder is expanded infinitely and behaves as 2-plane,

[5]In the sufficiently high temperature region such as $1/T < m_{\mathrm{A}}$, B would be proportinal to m_{A}^2.

[6]Unlike in the Lorentz type gauge fixing, the phase structure of TQFT sector would not be retained in MAG once perturbative deformation part is included. Therefore the deconfining phase transition must be explained in different way from in the case of the Lorentz type.

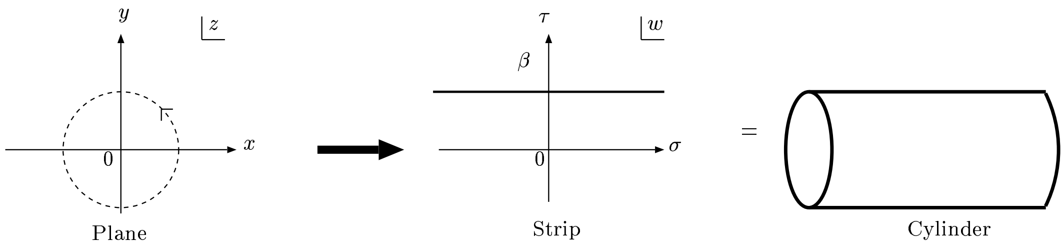

Figure 4: Thermal effect is realized as circumference of a cylinder or width of a strip. Conformal transformation $w = \beta/2\pi \ln z$ maps the z-plane onto the strip.

at high temperature cylinder collapses and behaves as 1D line at sufficiently high temperature. This implies that it is so simple and geometric to study the thermal effect on the linear potential and the deconfining phase transition. All we have to do is to investigate the behavior of topological objects on a cylinder. This is the great advantage of our deconfinement scenario.

Thermal effect on the instantons was discussed in ref. [24]. At high temperature the cylinder radius constrains the instanton size and the behavior of these is changed as the temperature increases. Large-size instantons are suppressed and so instanton effects can be reliably calculated at sufficiently high temperature. That is, the dilute gas approximation is rather valid, and so the interaction of instantons should be suppressed. In fact, at sufficiently high temperature the instanton effect could be ignored.

However, it is difficult to deal with the instanton beyond the dilute gas approximation and to calculate concretely. Therefore we consider the case of a compact U(1) gauge theory i.e. compact QED without matter. In QCD, we must deal with instanton and so worry with the various problem like infrared problem. But there is no problem in QED as such. If the gauge group U(1) is compact, abelian monopoles and the confining phase exist. In this case, TQFT sector is an O(2) NLSM on the cylinder and the linear potential is induced by the vortices of an O(2) NLSM, which should be interpreted as abelian monopoles from 4 dimensional view. Vortices are realized by the globally neutral Coulomb gas on the cylinder (on 2-plane at zero temperature). It was showed in ref. [10] that the confining-deconfining phase transition at zero temperature can be described by BKT phase transition at certain critical coupling. Using the expression of the propagator on the cylinder we could find the thermal effect on this phase transition. In the next section, we will discuss in detail.

Order of phase transition. Here, we comment on the order of the deconfining phase transition. There are two possibilities of the deconfinement mechanism. One is that the string tension vanishes discontinuously because of the vanishing of topological object effect as is discussed above. This corresponds to the first order phase transition. Thus, the both possibilities of the deconfining phase transition seems to exist. Another is that the string tension continuously vanishes at sufficiently high temperature because of e^{-S_1} dumping or the T-dependence of B. This case would correspond to the second order transition.

In general, it is considered that the order of deconfinement phase transition depends on the gauge group. The deconfinement of 4D SU(N) QCD without quarks can be related to the ferromagnetic-paramagnetic transition of $\mathbb{Z}_N$ spin model with the ferromagnetic interactions and $\mathbb{Z}_N$ symmetry plays a crucial role in the confinement problem. In particular, in the case of $N = 2$, it is related to 3D $\mathbb{Z}_2$ spin model (Ising), and the deconfinement transition is the second order. Also, in the case of $N = 3$ [25], it is related to 3D three-state Potts model, the phase transition of which is the first order one also expects that a deconfinement phase transition in an SU(3) theory, is of first order.

In the above scenario, it is unclear how we can explain the order of deconfinement transition. We could possibly explain it from the intense study of the behavior of instantons on reduced 2D theory [26].

4. Deconfining phase transition in compact QED_4

The phase structure of compact QED_4 at zero temperature has been worked out in ref. [10]. It has confining phase above the certain value of the coupling. Whether such a phase exists or not is decided due to the compactness of U(1). If U(1) is non-compact, such phase can not exist. In next subsection, we study the thermal effect on the confining phase.

4.1 Compact QED_4 at Finite Temperature

We consider the case of compact QED_4, which has the confining phase in the strong coupling region (UV region). In the similar way, its TQFT_4 sector becomes an O(2) NLSM_2, vortices of which induce the confining potential. The contribution of vortices to the partition function is described by 2D classical Coulomb gas. One of the difference between QCD and QED is that there exists no scale parameter (like Λ parameter in QCD) in QED and also no size parameter in vortex solutions of O(2) NLSM_2. Hence the treatment is rather simple and strict. When we consider the finite temperature case, the TQFT sector becomes O(2) NLSM_2 on the cylinder and vortices contribution could be described by the Coulomb gas on the cylinder.

The phase structure of 2D Coulomb gas is well known as shown figure 5. Here, we would like to consider the Coulomb gas on the cylinder. The propagator on the cylinder is given by

$$G(w-w') = -\frac{1}{4\pi}\ln\left|e^{\frac{2\pi}{\beta}w} - e^{\frac{2\pi}{\beta}w'}\right|^2 + \frac{1}{2\beta}\mathrm{Re}(w+w')\,, \tag{4.1}$$

where $w = \sigma + i\tau$ and τ is the imaginary time. We can find eq. (4.1) by constructing this propagator on the complex plane and then mapping it on a cylinder using

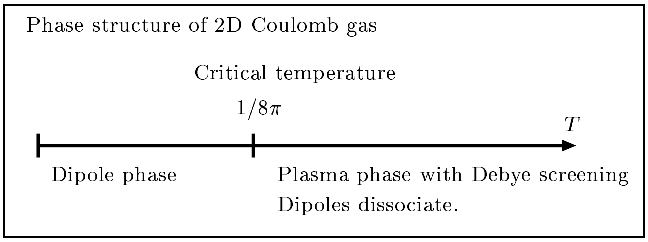

Figure 5: 2D Coulomb gas has two different phases. Over the critical temperature $1/8\pi$, the system is the plasma phase with Debye screening, and therefore mass gap exists. Below $1/8\pi$ it is the dipoles phase, in which Coulomb charges form dipoles. The system has a long range correlation and no mass gap.

conformal transformation.[7] Then, requiring that $G(w-w')$ is a function of $w-w'$, an additional term appears. The propagator eq. (4.1) is rewritten as

$$G(w-w') = -\frac{1}{4\pi}\ln\left|e^{\frac{2\pi}{\beta}(w-w')}-1\right|^2 + \frac{1}{2\beta}\mathrm{Re}(w-w')\,. \tag{4.2}$$

In the low temperature region ($\beta\to\infty$),

$$\begin{aligned} G(w-w') &\sim -\frac{1}{4\pi}\ln\left|\frac{2\pi}{\beta}(w-w')\right|^2 \\ &\sim -\frac{1}{2\pi}\ln|w-w'|\,. \end{aligned} \tag{4.3}$$

As is expected, this eq. (4.2) becomes usual 2D Coulomb potential, and behaves as the propagator on the plane. The constant term can be ignored because of the neutrality of the Coulomb gas. Next, we consider in the high temperature region ($\beta\to 0$). In the case of $\mathrm{Re}(w-w')>0$,

$$\begin{aligned} G(w-w') &\sim -\frac{1}{4\pi}\ln\left|e^{\frac{2\pi}{\beta}(w-w')}\right|^2 + \frac{1}{2\beta}\mathrm{Re}(w-w')\,, \\ &= -\frac{1}{2\beta}\mathrm{Re}(w-w')\,. \end{aligned} \tag{4.4}$$

In the case of $\mathrm{Re}(w-w')<0$,

$$\begin{aligned} G(w-w') &\sim -\frac{1}{4\pi}\ln|-1|^2 + \frac{1}{2\beta}\mathrm{Re}(w-w')\,, \\ &= \frac{1}{2\beta}\mathrm{Re}(w-w')\,. \end{aligned} \tag{4.5}$$

Therefore we find in the high temperature region ($\beta\to 0$)

$$G(w-w') \sim -\frac{1}{2\beta}|\mathrm{Re}(w-w')| = -\frac{1}{2\beta}|\sigma-\sigma'|\,. \tag{4.6}$$

[7] Another derivation of (4.1) is given in appendix A

This is 1D Coulomb potential. As is expected, the propagator behaves as on a line. Note that the factor $1/\beta$ appears. This factor does not appear if we naively start from 1D Coulomb gas. The origin consists in the finite cylinder radius. This factor is very important in the later discussion.

Let us discuss the behavior of Coulomb gas. It behaves as 1D Coulomb gas and its partition function is given by

$$Z_{1C} = \sum_{n=0}^{\infty} \frac{\zeta^n}{(n!)^2} \int \prod_{j=1}^{n} d\sigma_j \exp\left[(2\pi)^2 \frac{1}{\beta g^2} \sum_{i,j} Q_i Q_j |\sigma_i - \sigma_j|\right], \qquad \zeta \equiv \mathrm{e}^{-S_{(1)}}, \quad (4.7)$$

where $S_{(1)}$ is a single vortex action. Note that the temperature of this Coulomb gas system is $\theta \equiv \beta g^2 = g^2/T$. The 1D Coulomb gas is exactly solvable [27]. Its behavior in large θ and small θ is well known. Its phase structure is very similar with the 2D Coulomb gas. For small θ the 1D Coulomb gas behaves as the gas of free "molecules", made up of $+-$ charges pairs bound together. On the contrary, for large θ, the charges are completely deconfined, forming an electrically neutral "plasma" of $2n$ free particles. Now we are considering the high temperature region ($\beta \to 0$, that is $T \to \infty$) i.e. the small θ, and the 1D Coulomb gas behaves as a gas of free "molecules". This implies that the linear potential always vanishes irrelevantly to the definite coupling constant if the temperature is sufficiently large. Of course, if the coupling g^2 becomes much larger than T, that is we consider such an energy scale where g^2 becomes too large, the theory is confined again. The relative measurement between the coupling g^2 and the physical temperature T is important to decide whether confining or deconfining.

As a result, we could say that compact QED_4 is deconfined at the sufficiently high temperature. This nicely corresponds to the lattice compact U(1) gauge theory [25, 28]. Its phase diagram is shown in figure 6. Therefore we conclude that our scenario describes the confinement-deconfinement in compact QED_4 at zero and finite temperature very well.

4.2 Abelian dominance and deconfining transition

In this section, using the previous results and the abelian dominance, we discuss the deconfining phase transition of an $\mathrm{SU}(N)$ QCD_4.

In MAG, it is expected that off-diagonal gluons are massive and diagonal gluons are massless and that the diagonal component dominate in a sense of Wilsonian effective action. It is confirmed by Monte Carlo simulation on a lattice that off-diagonal gluons are massive [29, 30]. As an analytical derivation has been given at least in the TQFT sector based on the PS dimensional reduction to the coset G/H NLSM_2 [2]. Once this fact would be assumed, we could deal an $\mathrm{SU}(N)$ QCD_4 as an $\mathrm{U}(1)^{N-1}$ abelian gauge theory in the low energy region, which is $N-1$ copies of compact QED_4. That is, the low energy effective theory becomes an abelian gauge theory.

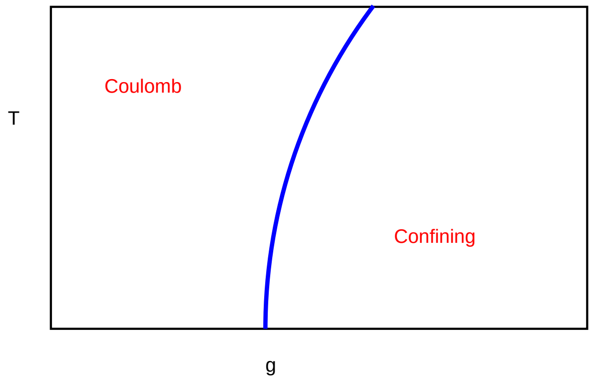

Figure 6: Phase diagram of the lattice compact U(1) gauge theory in 4D. The horizontal line is the coupling constant and the vertical one is the temperature.

Therefore we can apply the result of QED_4 in previous subsection. This scenario has been proposed in ref. [12], which also can be applied to the finite temperature case. This $U(1)^{N-1}$ gauge theory which is derived under the assumption of abelian dominance, has asymptotic freedom in contrast to usual (compact) QED_4. Hence the coupling constant becomes large in the IR region as the same as QCD_4. It is clear that the thermal effect could cause the deconfinement phase transition at sufficiently high temperature from the result of previous subsection if we begin with this low energy effective theory. Hence in this scenario our deconfinement mechanism works well.

5. Conclusions and discussions

We have investigated deconfining transition of QCD_4 and QED_4 at finite temperature. In our quark confinement scenario, using PS reduction the analysis is reduced to 2D theory on the cylinder, in which the thermal effect is realized through the radius of cylinder. Therefore, it is very simple and geometrical to investigate thermal effect on the deconfining transition. In order to discuss the deconfinement it is enough to study the behavior of topological objects on the cylinder with several radii. In particular, we could investigate the phase structure of the compact U(1) gauge theory in "*continuum*". This result agree with the result of lattice.

Taking high temperature limit is equivalent to compactification on a small circle S^1, and so we could discuss the relation between different dimensions, for example QCD_4 and QCD_5. 5D theory has been very interesting since it is widely discussed in the context of theory with extra dimensions. It is interesting work to investigate the connection between 5D and 4D QCD with the procedure in this paper. Also, our method might be applied to the system with Higgs fields such as Georgi-Glashow model. In ref. [31], in which the Georgi-Glashow model in 3D at finite temperature is discussed, at high temperature the system become 2D theory and BKT phase transition appears.

It is interesting to consider more general case $G = \mathrm{SU}(N)$ rather than $G = \mathrm{SU}(2)$ that is mainly discussed in this paper. If $N \geq 3$, there is the possibility of choosing the coset. The simplest example is a maximal torus $\mathrm{SU}(N)/\,\mathrm{U}(1)^{N-1}$. In this case, $N-1$ kinds of monopoles exist. However, a single monopole is enough to induce the linear potential, and so $\mathrm{SU}(N)/(\mathrm{SU}(N-1)\times \mathrm{U}(1)) \cong \mathbb{C}P^{N-1}$ is also possible. The confinement needs at least one $\mathrm{U}(1)$ factor in the above scenario. In $\mathbb{C}P^{N-1}$ case, the similar argument with the case of $G = \mathrm{SU}(2)$ seems to be possible [15]. The large N behavior of $\mathbb{C}P^{N-1}$ model is well understood. Therefore we might investigate the large N behavior of QCD_4 with our method of the perturbative deformation.

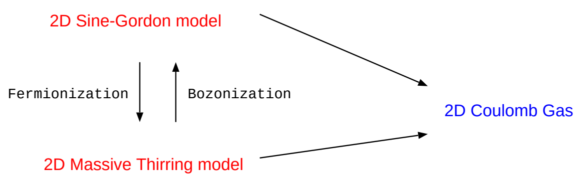

Figure 7: 2D Coulomb gas is intimately related to 2D sine-Gordon model and 2D massive Thirring model.

In this paper, we investigated the phase structure of the compact $\mathrm{U}(1)$ gauge theory at sufficiently high temperature and strong coupling using the behavior of the 1D Coulomb gas. The intimate relation among the classical Coulomb gas, massive Thirring model and sine-Gordon model at zero temperature is well known as indicated in figure 7. This relation at finite temperature is discussed in refs. [32, 33]. In particular, we could discuss the phase structure of the compact $\mathrm{U}(1)$ gauge theory in terms of sine-Gordon model on the cylinder instead of studying the classical Coulomb gas behavior [34]. The phase structure of sine-Gordon model at finite temperature is discussed in the ref. [35]. Also, it is attractive to include dynamical fermions and consider a finite chemical potential, chiral symmetry or flavor quantum number N_{f}.

The relation to the result of the dual Ginzburg-Landau theory is also interesting. In the case of $N = 3$ the critical temperature of the deconfinement has been estimated in ref. [36], based on the dual Ginzburg-Landau description.

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A. The propagator of a periodic boson on a cylinder

The action

$$S = \frac{1}{2}\int_0^\beta dx_0 \int dx_1\ \partial_\mu\phi\partial_\mu\phi \tag{A.1}$$

describes massless free boson on a cylinder.

The propagator is given by

$$D(x-y) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int \frac{dk_1}{2\pi}\mathrm{e}^{-ik\cdot(x-y)}\frac{1}{k^2+\mu^2}\,,$$
$$= \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int \frac{dk_1}{2\pi}\mathrm{e}^{-ik\cdot(x-y)}\int_0^\infty ds\;\mathrm{e}^{-s(k^2+\mu^2)}\,, \tag{A.2}$$

where $k^2 = k_0^2 + k_1^2, k_0 = 2\pi n/\beta$ ($n \in \mathbb{Z}$) is Matsubara frequency. In the above calculation we introduced the mass term μ in order to avoid the infrared divergence. The above expression (A.2), using the modified Bessel function $K_0(z)$, is rewritten as

$$D(x-y) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} K_0\left(\mu\sqrt{(x_0-y_0-n\beta)^2+(x_1-y_1)^2}\right). \tag{A.3}$$

In the limit $\mu \to 0$, $D(x-y)$ becomes

$$D(x-y) \sim -\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\ln\left(\mu\sqrt{(x_0-y_0-n\beta)^2+(x_1-y_1)^2}\right),$$
$$= -\frac{1}{2\pi}\ln\left(\mu\beta\sqrt{\cosh\left(\frac{2\pi}{\beta}(x_1-y_1)\right)-\cos\left(\frac{2\pi}{\beta}(x_0-y_0)\right)}\right). \tag{A.4}$$

Here, we introduce the complex coordinates

$$w = x_1 + ix_0\,,\quad (\bar{w} = x_1 - ix_0)\,,\qquad w' = y_1 + iy_0\,,\quad (\bar{w}' = y_1 - iy_0)\,, \tag{A.5}$$

and rewrite $D(x-y)$ by the use of these coordinates (A.5). Simple calculation leads to the following expression,

$$D(x-y) = -\frac{1}{4\pi}\ln\left|\mathrm{e}^{\frac{2\pi}{\beta}w}-\mathrm{e}^{\frac{2\pi}{\beta}w'}\right|^2 + \frac{1}{2\beta}\mathrm{Re}(w+w') - \frac{1}{4\pi}\ln\left(\frac{1}{2}(\mu\beta)^2\right). \tag{A.6}$$

The last term in the eq. (A.6) implies the infrared divergence. But this term can be ignored because of the neutral condition of Coulomb gas. Thus, we obtained the propagator (4.1). Also, we can relate the sine-Gordon model on the cylinder with the classical Coulomb gas on the cylinder, using the propagator (4.1).

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